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Journal of Approximation Theory 137 (2005) 250–263
Journal of Approximation Theory
www.elsevier.com/locate/jat

Classical and approximate sampling theorems; studies in the $L^P(\mathbb{R})$ and the uniform norm

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Received 4 March 2005; accepted 7 July 2005

Communicated by Hans G. Feichtinger

Oved Shisha (1932–1998) in Memoriam—The Founder of the Journal of Approximation Theory [1]

Abstract

The approximate sampling theorem with its associated aliasing error is due to J.L. Brown (1957). This theorem includes the classical Whittaker–Kotel'nikov–Shannon theorem as a special case. The converse is established in the present paper, that is, the classical sampling theorem for $f \in B^p_{\pi w}$, $1 \leqslant p < \infty$, $w > 0$, implies the approximate sampling theorem. Consequently, both sampling theorems are fully equivalent in the uniform norm.

Turning now to $L^p(\mathbb{R})$-space, it is shown that the classical sampling theorem for $f \in B^p_{\pi w}$, $1 < p < \infty$ (here $p = 1$ must be excluded), implies the $L^p(\mathbb{R})$-approximate sampling theorem with convergence in the $L^p(\mathbb{R})$-norm, provided that f is locally Riemann integrable and belongs to a certain class Λ^p. Basic in the proof is an intricate result on the representation of the integral $\int_{\mathbb{R}} |f(u)|^p\, du$ as the limit of an infinite Riemann sum of $|f|^p$ for a general family of partitions of $\mathbb{R}$; it is related to results of O. Shisha et al. (1973–1978) on simply integrable functions and functions of bounded coarse variation on $\mathbb{R}$. These theorems give the missing link between two groups of major equivalent theorems; this will lead to the solution of a conjecture raised a dozen years ago.
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MSC: 94A20; 94A12; 41A35; 41A80

Keywords: Sampling theory; Signal theory; Approximation by discrete operators

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[1] see: *Dedication*, J. Approx. Theory 86 (1) (1996) 1–12.

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doi:10.1016/j.jat.2005.07.011

1. Introduction

For $1 \leqslant p < \infty$, consider the space

$$F^p := \{f \in L^p(\mathbb{R}) \cap C(\mathbb{R});\ f^\wedge \in L^1(\mathbb{R}) \cap L^{p'}(\mathbb{R})\},$$

with $1/p + 1/p' = 1$, and for $w > 0$ the Bernstein space of functions bandlimited to $[-\pi w, \pi w]$,

$$B^p_{\pi w} := \{f \in L^p(\mathbb{R}) \cap C(\mathbb{R});\ \operatorname{supp} f^\wedge \subset [-\pi w, \pi w]\}.$$

Here $f^\wedge(v) := (1/\sqrt{2\pi}) \int_{\mathbb{R}} f(u) e^{-ivu}\, du$ denotes the Fourier transform of f, in the case $p > 2$ to be understood in the distributional sense. The condition $f^\wedge \in L^{p'}(\mathbb{R})$ in the definition of F^p is always satisfied for $1 \leqslant p \leqslant 2$.

Theorem A (*Classical Sampling Theorem*). *Let $f \in B^p_{\pi w}$, $1 \leqslant p < \infty$, then*

$$f(t) = \sum_{k \in \mathbb{Z}} f\left(\frac{k}{w}\right) \operatorname{sinc}(wt - k) \quad (t \in \mathbb{R}), \tag{1}$$

the series being absolutely and uniformly convergent on $\mathbb{R}$.

In the following the series in (1) will be referred to as sampling series of f. For the history of this theorem see e.g., [4,3].

If f is not bandlimited, Theorem A is no longer true, but it may hold at least in the limit for $w \to \infty$. Indeed, one has (see [4,8, pp. 95, 118–122] and literature cited there):

Theorem B (*Approximate (or Generalized) Sampling Theorem*). *Let $f \in F^p$, $1 \leqslant p < \infty$, and let*

$$(R_w f)(t) := \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} \left(1 - e^{-it2\pi wn}\right) \int_{(2n-1)\pi w}^{(2n+1)\pi w} f^\wedge(v) e^{itv}\, dv).$$

Then

$$f(t) = \sum_{k \in \mathbb{Z}} f\left(\frac{k}{w}\right) \operatorname{sinc}(wt - k) + (R_w f)(t) \quad (t \in \mathbb{R}) \tag{2}$$

together with the error estimate

$$|(R_w f)(t)| \leqslant \sqrt{\frac{2}{\pi}} \int_{|v| \geqslant \pi w} |f^\wedge(v)|\, dv \quad (t \in \mathbb{R}). \tag{3}$$

In particular, one has

$$f(t) = \lim_{w \to \infty} \sum_{k \in \mathbb{Z}} f\left(\frac{k}{w}\right) \operatorname{sinc}(wt - k) \tag{4}$$

uniformly for $t \in \mathbb{R}$.

Observe that $|(R_w f)(t)|$ is the so-called aliasing error, first treated in a monumental paper of de La Vallée Poussin in 1908 (see [2] pp. 65–156 for a reproduction of his paper, and pp. 421–453 for a commentary by Butzer–Stens).

Clearly, assertion (2) of the Approximate Sampling Theorem includes in view of (3) the Classical Sampling Theorem. Here we ask for the converse: Does the Classical Sampling Theorem imply the Approximate Sampling Theorem? The result is Theorem 1, the proof of which will follow in Section 2.

Theorem 1. *Under the given hypothesis that* $f \in F^p$, $1 \leqslant p < \infty$, *Theorem A implies the approximate sampling theorem, namely Theorem B.*

In particular, Theorems A and B are equivalent, each implies the other without any additional conditions.

In Section 3 we will consider an L^p-version of the approximate sampling theorem, thus the L^p-counterpart of (4), namely:

Theorem 2. *Under the hypothesis* $f \in \Lambda^p \cap R_{\mathrm{loc}}$, $1 < p < \infty$, *the classical sampling theorem, specifically Theorem A for* $f \in B^p_{\pi w}$, $1 < p < \infty$, *yields the* L^p*-approximate sampling theorem in the form*

$$\lim_{w\to\infty} \left\| f(t) - \sum_{k\in\mathbb{Z}} f\left(\frac{k}{w}\right) \operatorname{sinc}(wt-k) \right\|_{L^p(\mathbb{R})} = 0. \tag{5}$$

Here Λ^p is a suitable subspace of $L^p(\mathbb{R})$ (Definition 4) guaranteeing that the series is convergent for all $t \in \mathbb{R}$, and R_{loc} denotes the space of all locally Riemann integrable functions.

Observe that an approximate sampling theorem, namely that the limit relation (5) holds under suitable conditions (see below), was established in [16,5,17,1]. However, it was not shown in these papers that this result does indeed follow explicitly from the classical sampling theorem.

We have actually shown that in $C(\mathbb{R})$-space the sampling theorem for band-limited functions is fully equivalent to the approximate sampling theorem for not necessarily band-limited functions.

Each of these theorems has long been known to embody basic principles of sampling and interpolation theory. The theorems can now be seen as saying the same thing, that is, they are two different manifestations of the same underlying scientific principle. However, this underlying principle is not completely understood at the present time; nevertheless, the equivalence of Theorems A and B provides a basic step towards our better understanding of this underlying principle.

The error entailed (see (3), (20)) is an integral part of an approximate sampling theorem; for the error in $C(\mathbb{R})$-space recall, e.g., [3,4], and in case of L^p-spaces see [1].

2. The approximate sampling theorem in the uniform norm

Proof of Theorem 1. First assume $p \geqslant 2$. Since $f \in F^p$ the Fourier inversion formula gives

$$\begin{aligned} f(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f^\wedge(v) e^{ivt}\, dv \\ &= \frac{1}{\sqrt{2\pi}} \int_{|v|\leqslant \pi w} f^\wedge(v) e^{ivt}\, dv + \frac{1}{\sqrt{2\pi}} \int_{|v|>\pi w} f^\wedge(v) e^{ivt}\, dv =: f_1(t) + f_2(t), \end{aligned} \tag{6}$$

say. Now, $f^\wedge \in L^{p'}(\mathbb{R})$ implies $f_1 \in B^p_{\pi w}$, and so the classical sampling theorem (Theorem A) can be applied to it, giving

$$f_1(t) = \sum_{k=-\infty}^{\infty} f_1\left(\frac{k}{w}\right) \operatorname{sinc}(wt-k) =: (S_w f_1)(t). \tag{7}$$

Concerning the sampling series of the function f_2 we need the following lemma, the proof of which will follow.

Lemma 3. *Let f_2 be defined as above, then the sampling series*

$$(S_w f_2)(t) := \sum_{k=-\infty}^{\infty} f_2\left(\frac{k}{w}\right) \operatorname{sinc}(wt-k) = \lim_{N\to\infty} \sum_{k=-N}^{N} f_2\left(\frac{k}{w}\right) \operatorname{sinc}(wt-k) \tag{8}$$

is convergent for all $t \in \mathbb{R}$ and can be rewritten as

$$(S_w f_2)(t) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}}{}' e^{-it2\pi wn} \int_{(2n-1)\pi w}^{(2n+1)\pi w} f^\wedge(v) e^{itv}\, dv \quad (t \in \mathbb{R}).$$

Proceeding with the proof of Theorem 1, it follows from (6)–(8) that

$$\begin{aligned} f(t) &= \sum_{k=-\infty}^{\infty} f_1\left(\frac{k}{w}\right) \operatorname{sinc}(wt-k) + f_2(t) \\ &= \sum_{k=-\infty}^{\infty} \left[f_1\left(\frac{k}{w}\right) + f_2\left(\frac{k}{w}\right)\right] \operatorname{sinc}(wt-k) - \left\{(S_w f_2)(t) - f_2(t)\right\} \\ &= \sum_{k=-\infty}^{\infty} f\left(\frac{k}{w}\right) \operatorname{sinc}(wt-k) - \left\{(S_w f_2)(t) - f_2(t)\right\}. \end{aligned} \tag{9}$$

As to the term in curly brackets, we obtain by our lemma and the definition of f_2 that

$$\begin{aligned} \left\{(S_w f_2)(t) - f_2(t)\right\} &= \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}}{}' e^{-it2\pi wn} \int_{(2n-1)\pi w}^{(2n+1)\pi w} f^\wedge(v) e^{itv}\, dv \\ &\quad - \frac{1}{\sqrt{2\pi}} \int_{|v|>\pi w} f^\wedge(v) e^{ivt}\, dv \\ &= \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \left(e^{-it2\pi wn} - 1\right) \int_{(2n-1)\pi w}^{(2n+1)\pi w} f^\wedge(v) e^{itv}\, dv \\ &= -\,(R_w f)(t). \end{aligned}$$

Inserting this into (9) gives (2), and an obvious estimate of the remainder $(R_w f)(t)$ yields the error bound in (3).

If $1 \leqslant p < 2$, then $p' > 2$, and the assumption $f^\wedge \in L^{p'}(\mathbb{R})$ does not in general imply $f_1 \in L^p(\mathbb{R})$. So f_1 does not necessarily belong to $B^p_{\pi w}$ and the above arguments cannot be used.

However, $f \in L^p(\mathbb{R})$ together with $f^\wedge \in L^1(\mathbb{R})$ implies $f \in L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$ which in turn implies $f \in L^2(\mathbb{R})$. So $F^p \subset F^2$ for $1 \leqslant p < 2$, and the proof of Theorem 1 is complete. □

Proof of Lemma 3. For fixed $t \in \mathbb{R}$ let g_t be the $2\pi w$-periodic extension of the function $v \mapsto e^{ivt}$ from the interval $(-\pi w, \pi w]$ to the whole real axis $\mathbb{R}$, i.e.,

$$g_t(v) = e^{it(v-2\pi wn)} \quad \big(v \in ((2j-1)\pi w, (2j+1)\pi w];\ j \in \mathbb{Z}\big). \tag{10}$$

The Fourier coefficients of g_t are $\operatorname{sinc}(wt-k)$, $k \in \mathbb{Z}$, and since g_t is of bounded variation, its Fourier series converges at each point of continuity to g_t with uniformly bounded partial sums (see [18, p. 28]). Hence we have

$$g_t(v) = \sum_{k=-\infty}^{\infty} \operatorname{sinc}(wt-k) e^{ikv/w} \quad (v \neq (2j+1)\pi w;\ j \in \mathbb{Z}), \tag{11}$$

$$\sum_{k=-N}^{N} \operatorname{sinc}(wt-k) e^{ikv/w} \leqslant C \quad (v \in \mathbb{R};\ N \in \mathbb{N}). \tag{12}$$

Now one has for the series in question by (11)

$$\begin{aligned}
&\lim_{N\to\infty} \sum_{k=-N}^{N} f_2\left(\frac{k}{w}\right) \operatorname{sinc}(wt-k) \\
&\quad = \lim_{N\to\infty} \sum_{k=-N}^{N} \left\{ \frac{1}{\sqrt{2\pi}} \int_{|v|>\pi w} f^\wedge(v) e^{ikv/w}\, dv \right\} \operatorname{sinc}(wt-k) \\
&\quad = \lim_{N\to\infty} \left\{ \frac{1}{\sqrt{2\pi}} \int_{|v|>\pi w} f^\wedge(v) \sum_{k=-N}^{N} e^{ikv/w} \operatorname{sinc}(wt-k)\, dv \right\} \\
&\quad = \frac{1}{\sqrt{2\pi}} \int_{|v|>\pi w} f^\wedge(v) \lim_{N\to\infty} \sum_{k=-N}^{N} e^{ikv/w} \operatorname{sinc}(wt-k)\, dv \\
&\quad = \frac{1}{\sqrt{2\pi}} \int_{|v|>\pi w} f^\wedge(v) g_t(v)\, dv, \qquad (13)
\end{aligned}$$

the interchange of the limit and the integral being justified by Lebesgue's dominated convergence theorem, noting that in view of (12),

$$\left| f^\wedge(v) - \sum_{k=-N}^{N} \operatorname{sinc}(wt-k) e^{ikv/w} \right| \leqslant C |f^\wedge(v)| \in L^1(\mathbb{R}) \quad (N \in \mathbb{N}).$$

This proves the first part of the lemma. The second part now follows easily because in view of (13)

$$
\begin{aligned}
(S_w f_2)(t) &= \frac{1}{\sqrt{2\pi}} \int_{|v|>\pi w} f^{\wedge}(v) g_t(v)\, dv \\
&= {\sum_{n\in\mathbb{Z}}}' \frac{1}{\sqrt{2\pi}} \int_{(2n-1)\pi w}^{(2n+1)\pi w} f^{\wedge}(v) g_t(v)\, dv \\
&= {\sum_{n\in\mathbb{Z}}}' \frac{1}{\sqrt{2\pi}} \int_{(2n-1)\pi w}^{(2n+1)\pi w} f^{\wedge}(v) e^{it(v-2\pi w n)}\, dv,
\end{aligned}
$$

the last equation being valid by (10). □

3. Sampling series in $L^p(\mathbb{R})$-space

3.1. The space Λ^p

When studying sampling series in L^p-spaces there arises the problem that the series $S_w f$ of (1) of an L^p-function f may be divergent. Thus for the function $f_0(t) := 1$ for $t \in \mathbb{Z}$ and $= 0$ otherwise with $\|f_0\|_{L^p(\mathbb{R})} = 0$, $1 \leqslant p < \infty$, one has $\|S_w f_0 - f_0\|_{L^p(\mathbb{R})} = \infty$ provided $w = m/n$ is rational (see [1, Example 37]). So we have to restrict the matter to a suitable subspace which guarantees that the series $S_w f$ is convergent for each signal function f in this space. To this end we introduced the space Λ^p in [1].

Definition 4. (a) A sequence $\Sigma := (x_j)_{j\in\mathbb{Z}} \subset \mathbb{R}$ is called an *admissible* (*or allowable*) *partition of* $\mathbb{R}$ or an *admissible sequence*, if it satisfies the inequalities

$$0 < \underline{\Delta}_\Sigma \equiv \underline{\Delta} := \inf_{j\in\mathbb{Z}} (x_j - x_{j-1}) \leqslant \overline{\Delta}_\Sigma \equiv \overline{\Delta} := \sup_{j\in\mathbb{Z}} (x_j - x_{j-1}) < \infty; \tag{14}$$

$\underline{\Delta}$ and $\overline{\Delta}$ are called the *lower and upper mesh size* of Σ, respectively.

(b) Let $\Sigma := (x_j)_{j\in\mathbb{Z}}$ be an admissible partition of $\mathbb{R}$, and let $\Delta_j := x_j - x_{j-1}$. The discrete $l^p(\Sigma)$-*norm* of a sequence of function values f_Σ on the partition Σ of a function $f: \mathbb{R} \to \mathbb{C}$ is defined for $1 \leqslant p < \infty$ by

$$\|f_\Sigma\|_{l^p(\Sigma)} \equiv \|f\|_{l^p(\Sigma)} := \left\{ \sum_{j\in\mathbb{Z}} |f(x_j)|^p \Delta_j \right\}^{1/p}.$$

(c) The space Λ^p for $1 \leqslant p < \infty$ is defined by

$$\Lambda^p := \{ f \in M(\mathbb{R});\ \|f\|_{l^p(\Sigma)} < \infty \text{ for each admissible sequence } \Sigma \},$$

$M(\mathbb{R})$ being the space of all Lebesgue measurable and bounded functions $f: \mathbb{R} \to \mathbb{C}$.

$\|f_\Sigma\|_{l^p(\Sigma)} = \|f\|_{l^p(\Sigma)}$ can also be regarded as a semi-norm on Λ^p. Note that any admissible partition $(x_j)_{j\in\mathbb{Z}}$ is strictly monotone increasing with

$$\lim_{j\to-\infty} x_j = -\infty, \qquad \lim_{j\to\infty} x_j = \infty.$$

For a fixed partition $\Sigma = (x_j)_{j\in\mathbb{Z}}$, the $l^p(\Sigma)$-norm of the sequence $\big(f(x_j)\big)_{j\in\mathbb{Z}}$ can be estimated from above and below by the usual $l^p(\mathbb{Z})$-norm; indeed,

$$\underline{\Delta}^{1/p}\left\{\sum_{j\in\mathbb{Z}}|f(x_j)|^p\right\}^{1/p} \leqslant \left\{\sum_{j\in\mathbb{Z}}|f(x_j)|^p\Delta_j\right\}^{1/p} \leqslant \overline{\Delta}^{1/p}\left\{\sum_{j\in\mathbb{Z}}|f(x_j)|^p\right\}^{1/p}. \tag{15}$$

If one, however, deals with sequences $(\Sigma_n)_{n\in\mathbb{N}}$ or families $(\Sigma_\rho)_{\rho\in\mathbb{A}}$ of partitions, as will be the present case, the use of the $l^p(\Sigma)$-norm often results in estimates with constants that are independent of the parameter n or ρ, e.g., Theorem 7.

For a function $f \in \Lambda^p$, $1 < p < \infty$, the series $S_w f$ is absolutely convergent; indeed, one has by Hölder's inequality with $1/p + 1/p' = 1$ (see [4, p. 18])

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\left|f\left(\frac{k}{w}\right)\operatorname{sinc}(wt-k)\right| &\leqslant \left\{\sum_{k\in\mathbb{Z}}\left|f\left(\frac{k}{w}\right)\right|^p\right\}^{1/p}\cdot\left\{\sum_{k\in\mathbb{Z}}\left|\operatorname{sinc}(wt-k)\right|^{p'}\right\}^{1/p'}\\ &< w^{1/p}\left\{\sum_{k\in\mathbb{Z}}\left|f\left(\frac{k}{w}\right)\right|^p\frac{1}{w}\right\}^{1/p}\cdot p \equiv w^{1/p}\,\|f\|_{l^p(\Sigma)}\cdot p<\infty.\end{aligned}$$

Instead of the space Λ^p, Rahman and Vértesi [16] considered the space $\mathfrak{F}^p \subset L^p(\mathbb{R})$ consisting of those functions $f \in M(\mathbb{R})$ for which there exists a $\eta > 0$ with

$$f(x) = \mathcal{O}\big((1+|x|)^{-1/p-\eta}\big) \quad (x \to \pm\infty).$$

Another possible subspace of $L^p(\mathbb{R})$ is

$$\begin{aligned}\Omega^p := \big\{&f \in M(\mathbb{R});\ |f(t)| \leqslant g(t),\ t \in \mathbb{R},\ \text{for } g \in L^p(\mathbb{R})\\ &\text{with } g \text{ non-negative, even, non-increasing on } [0,\infty)\big\},\end{aligned}$$

which was used by Fang [5]. In [1] it was shown that $\mathfrak{F}^p$ and Ω^p as well as the spaces $W^r(L^p(\mathbb{R}))\cap C(\mathbb{R})$, $r \in \mathbb{N}$, where

$$W^r(L^p(\mathbb{R})) := \big\{f \in L^p(\mathbb{R});\ f(t) = \varphi(t) \text{ a.e.}, \varphi \in AC^r_{\text{loc}}(\mathbb{R}), \varphi^{(r)} \in L^p(\mathbb{R})\big\}$$

is the Sobolev space, are actually proper linear subspaces of our Λ^p.

In order to study the space Λ^p in more detail we now introduce the function $f^*_\Sigma : \mathbb{R} \to [0,\infty)$, defined by

$$f^*_\Sigma(t) := \sup_{u\in[x_{j-1},x_j]}|f(u)|, \quad t \in (x_{j-1}, x_j], \quad j \in \mathbb{Z},$$

Σ being an admissible sequence. The function f^*_Σ is an "upper encasing" step function majorizing f. It plays a similar role as do the majorizing functions occurring in the definitions of the spaces $\mathfrak{F}^p$ and Ω^p, respectively, in [16] or [5], although it is not necessarily non-increasing.

Firstly, f^*_Σ is used to give an equivalent characterization of the space Λ^p.

Lemma 5. *The following assertions are equivalent for* $1 \leqslant p < \infty$:

(i) $f \in \Lambda^p$.
(ii) $f^*_\Sigma \in \Lambda^p$ *for each admissible sequence* Σ.

(iii) *For each admissible sequence* $\Sigma := (x_j)_{j\in\mathbb{Z}}$ *and each choice of points* $\xi_j \in [x_{j-1}, x_j]$, $j \in \mathbb{Z}$, *there holds*

$$\sum_{j\in\mathbb{Z}} |f(\xi_j)|^p (x_j - x_{j-1}) < \infty.$$

Proof. (i) $\Rightarrow$ (ii): Let $\Sigma := (y_j)_{j\in\mathbb{Z}}$ with mesh sizes $\overline{\Delta}_\Sigma$ and $\underline{\Delta}_\Sigma$, respectively. Then, for each $k \in \mathbb{Z}$ there exists a $t_k^* \in [y_{k-1}, y_k]$ such that

$$f_\Sigma^*(y_k) = \sup_{u\in[y_{k-1},y_k]} |f(u)| < |f(t_k^*)| + \frac{1}{k^2+1}.$$

It follows that

$$\left\{\sum_{k\in\mathbb{Z}} f_\Sigma^*(y_k)^p\right\}^{1/p} < \left\{\sum_{k\in\mathbb{Z}} |f(t_k^*)|^p\right\}^{1/p} + \left\{\sum_{k\in\mathbb{Z}} \left(\frac{1}{k^2+1}\right)^p\right\}^{1/p}.$$

Since the sequence $(t_k^*)_{k\in\mathbb{Z}}$ is not necessarily admissible, we split it up into two admissible ones, namely $(t_{2k+1}^*)_{k\in\mathbb{Z}}$ and $(t_{kj}^*)_{k\in\mathbb{Z}}$ with lower mesh size $\underline{\Delta}^* \geqslant \underline{\Delta}_\Sigma$, and upper mesh size $\overline{\Delta}^* \leqslant 3\overline{\Delta}_\Sigma$ for both. This yields by the definition of the space Λ^p,

$$\left\{\sum_{k\in\mathbb{Z}} f^*(y_k)^p\right\}^{1/p} < \left\{\sum_{k\in\mathbb{Z}} |f(t_{2k+1}^*)|^p + \sum_{k\in\mathbb{Z}} |f(t_{2k}^*)|^p\right\}^{1/p} + \left\{\sum_{k\in\mathbb{Z}} \left(\frac{1}{k^2+1}\right)^p\right\}^{1/p} < \infty. \tag{16}$$

Now, if $\Sigma' := (x_j)_{j\in\mathbb{Z}}$ is an arbitrary admissible sequence, then with $\Delta'_j := x_j - x_{j-1}$,

$$\sum_{j\in\mathbb{Z}} f^*(x_j)^p \Delta'_j = \sum_{k\in\mathbb{Z}} \sum_{x_j\in(y_{k-1},y_k]} f^*(x_j)^p \Delta'_j = \sum_{k\in\mathbb{Z}} f^*(y_k)^p \sum_{x_j\in(y_{k-1},y_k]} \Delta'_j. \tag{17}$$

Since $\sum_{x_j\in(y_{k-1},y_k]} \Delta'_j \leqslant \overline{\Delta}_{\Sigma'} + \overline{\Delta}_\Sigma$ the right-hand side of (17) is finite in view of (16). This gives (ii).

(ii) $\Rightarrow$ (iii): One has for each admissible $\Sigma := (x_j)_{j\in\mathbb{Z}}$ and arbitrary $\xi_j \in [x_{j-1}, x_j]$,

$$\sum_{j\in\mathbb{Z}} |f(\xi_j)|^p (x_j - x_{j-1}) \leqslant \sum_{j\in\mathbb{Z}} f_\Sigma(x_j)^p (x_j - x_{j-1}).$$

This gives (ii) $\Rightarrow$ (iii). The final step (iii) $\Rightarrow$ (i) follows with the special choice $\xi_j = x_j$. □

Some further properties of the space Λ^p can now be easily deduced. Property (b) gives a full proof of the representation of the integral $\int_{\mathbb{R}} |f(u)|^p\,du$ as a limit of an *infinite* Riemann sum of $|f|^p$ for a general family of partitions Σ_ρ. It is a generalization of Lemma 13 of Rahman–Vértesi [16] for the particular partition $\Sigma_w := \{\frac{k}{w}; k \in \mathbb{Z}\}$, the hypothesis there, namely $f \in \mathfrak{F}^p$, also being weakened to our $f \in \Lambda^p$. Of course, the corresponding result for the integral $\int_{-N}^{N} |f(u)|^p\,du$ with $f \in R_{\text{loc}}$ (which is also needed in the proof) follows obviously from the definition of the Riemann integral.

Lemma 6. *Let* $1 \leqslant p < \infty$.

(a) Λ^p *is a proper linear subspace of* $L^p(\mathbb{R})$.

(b) *Assume* $f \in \Lambda^p \cap R_{\mathrm{loc}}$, *and for each* $\rho > 0$ *let* $\Sigma_\rho := \left(x_j^{(\rho)}\right)_{j\in\mathbb{Z}}$ *be an admissible sequence with upper mesh size* $\overline{\Delta}^{(\rho)}$ *tending to zero for* $\rho \to \infty$. *Further, let* $\xi_j^{(\rho)}$, $j \in \mathbb{Z}$, *be arbitrary points with* $\xi_j^{(\rho)} \in [x_{j-1}{}^{(\rho)}, x_j^{(\rho)}]$. *Then*

$$\lim_{\rho\to\infty} \|f\|_{l^p(\Sigma_\rho)} \equiv \lim_{\rho\to\infty} \left\{ \sum_{j\in\mathbb{Z}} \left| f\left(\xi_j^{(\rho)}\right)\right|^p \Delta_j^{(\rho)} \right\}^{1/p} = \left\{ \int_{-\infty}^{\infty} |f(u)|^p \, du \right\}^{1/p} \equiv \|f\|_{L^p(\mathbb{R})}. \tag{18}$$

Proof. Concerning (a), Λ^p is obviously a linear manifold, and the integrability of $|f|^p$ is a consequence of Lemma 5, since

$$\int_{-\infty}^{\infty} |f(x)|^p \, dx = \sum_{j=-\infty}^{\infty} \int_j^{j+1} |f(x)|^p \, dx \leqslant \sum_{j=-\infty}^{\infty} f^*(j)^p < \infty,$$

where $f^* := f^*_{\mathbb{Z}}$ is the encasing function of Lemma 5 for $\Sigma = \mathbb{Z}$.

As to part (b), first one obtains similar to the proof of Lemma 5, (i) $\Rightarrow$ (ii),

$$\begin{aligned} \sum_{|j|\geqslant N} |f(\xi_j)|^p \Delta_j &\leqslant \sum_{|k|\geqslant N} \sum_{\xi_j \in (k-1,k]} |f(\xi_j)|^p \Delta_j \\ &= \sum_{|k|\geqslant N} f^*(k)^p \sum_{\xi_j\in(k-1,k]} \Delta_j \leqslant (2\overline{\Delta}+1) \sum_{|k|\geqslant N} f^*(k)^p. \end{aligned} \tag{19}$$

Now, one has for arbitrary $N \in \mathbb{N}$,

$$\begin{aligned} \left| \|f\|^p_{l^p(\Sigma_\rho)} - \|f\|^p_{L^p(\mathbb{R})} \right| &= \left| \sum_{j\in\mathbb{Z}} \left| f\left(\xi_j^{(\rho)}\right)\right|^p \Delta_j^{(\rho)} - \int_{-\infty}^{\infty} |f(u)|^p \, du \right| \\ &\leqslant \left| \sum_{|j|<N} \left| f\left(\xi_j^{(\rho)}\right)\right|^p \Delta_j^{(\rho)} - \int_{-N}^{N} |f(u)|^p \, du \right| \\ &\quad + \sum_{|j|\geqslant N} \left| f\left(\xi_j^{(\rho)}\right)\right|^p \Delta_j^{(\rho)} + \int_{|u|\geqslant N} |f(u)|^p \, du \\ &=: A_1 + A_2 + A_3, \end{aligned}$$

say. Assume now that $\rho_0 > 0$ is such that $\overline{\Delta}^{(\rho)} \leqslant 1$ for all $\rho \geqslant \rho_0$, and let $\varepsilon > 0$. In view of part (a), (19), and Lemma 5, we can choose N so large that $A_3 < \varepsilon$ and $A_2 < \varepsilon$ for all $\rho > \rho_0$. So we must show that A_1 tends to zero for $\rho \to \infty$.

For N as chosen above let m be the smallest integer such that $-N < x_m^{(\rho)}$, and let n be the largest integer such that $x_n^{(\rho)} < N$. It follows that

$$\sum_{|j|<N} \left|f\left(\xi_j^{(\rho)}\right)\right|^p \Delta_j^{(\rho)} = \sum_{j=m}^{n} \left|f\left(\xi_j^{(\rho)}\right)\right|^p \left(x_j^{(\rho)} - x_{j-1}^{(\rho)}\right)$$

$$= \left\{ \left|f\left(\xi_m^{(\rho)}\right)\right|^p \left(x_m^{(\rho)} - (-N)\right) \right.$$

$$\left. + \sum_{j=m+1}^{n} \left|f\left(\xi_j^{(\rho)}\right)\right|^p \left(x_j^{(\rho)} - x_{j-1}^{(\rho)}\right) + \left|f(N)\right|^p \left(N - x_n^{(\rho)}\right) \right\}$$

$$-\left|f\left(\xi_m^{(\rho)}\right)\right|^p \left(x_{m-1}^{(\rho)} - (-N)\right) - \left|f(N)\right|^p \left(N - x_n^{(\rho)}\right).$$

Here the term in curly brackets is a Riemannian sum of the integral $\int_{-N}^{N} |f(u)|^p \, du$ over the finite interval $[-N, N]$, whereas the two latter terms vanish for $\rho \to \infty$ in view of the boundedness of f, and

$$0 \leqslant \left(-N - x_{m-1}^{(\rho)}\right) \leqslant \left(x_m^{(\rho)} - x_{m-1}^{(\rho)}\right) \leqslant \overline{\Delta}^{(\rho)} = o(1) \quad (\rho \to \infty),$$

$$0 \leqslant \left(N - x_n^{(\rho)}\right) \leqslant \left(x_{n+1}^{(\rho)} - x_n^{(\rho)}\right) \leqslant \overline{\Delta}^{(\rho)} = o(1) \quad (\rho \to \infty).$$

Since f is locally Riemann integrable one obtains for each $N \in \mathbb{N}$,

$$\lim_{\rho \to \infty} \sum_{|j|<N} \left|f\left(\xi_j^{(\rho)}\right)\right|^p \Delta_j^{(\rho)} = \int_{-N}^{N} |f(u)|^p \, du,$$

which was left to be proved. □

There exists further equivalent characterizations of the function class Λ^p in terms of functions being simply integrable or being of bounded coarse variation, both over $\mathbb{R}$.

According to Haber and Shisha [7], a function $g : \mathbb{R} \to \mathbb{C}$ is said to be simply integrable over $\mathbb{R}$ provided for every $\varepsilon > 0$ there exist positive numbers B_1, B_2 and Δ such that

$$\left| \int_{-\infty}^{\infty} g(u) \, du - \sum_{j=-M}^{N} g(\xi_j)(x_j - x_{j-1}) \right| < \varepsilon$$

for every strictly increasing sequence $(x_j)_{j=-M}^{N}$ with $x_{-M} < -B_1$, $x_N > B_2$ and $x_j - x_{j-1} < \Delta$, as well as numbers $\xi_j \in [x_{j-1}, x_j]$. In this respect it was shown in [15] that a function $g \in R_{\text{loc}}$ satisfies condition (iii) of our Lemma 5 if and only if it is simply integrable over $\mathbb{R}$. Simple integrability in turn implies in the terminology of Lemma 6(b) that (cf. [12, Theorem 2])

$$\lim_{\rho \to \infty} \sum_{j \in \mathbb{Z}} g\left(\xi_j^{(\rho)}\right) \Delta_j^{(\rho)} = \int_{-\infty}^{\infty} g(u) \, du.$$

In view of Lemma 5, (i) $\Longleftrightarrow$ (iii) this means that $f \in \Lambda^p \cap R_{\text{loc}} \iff |f|^p$ is simply integrable over $\mathbb{R}$, and our Lemma 6(b) would also be a consequence of the results of Shisha et al. However, our proofs of Lemmas 5 and 6 are quite different, simpler and much shorter, their oscillation function $\mathcal{O}_j := \sup_{j-1\leqslant x<y\leqslant j} |f(y) - f(x)|$ with $\sum_{j\in\mathbb{Z}} \mathcal{O}_j < \infty$ being the rough counterpart of our $f^*_{\mathbb{Z}}$.

Another characterization of Λ^p can be given in terms of bounded coarse variation. Following [7] a function $g\colon \mathbb{R} \to \mathbb{C}$ is of coarse bounded variation over $\mathbb{R}$, $g \in BCV(\mathbb{R})$, if for every $\Delta > 0$

$$\sup \sum_j |g(x_j) - g(x_{j-1})| < \infty,$$

the supremum being taken over all strictly increasing finite or infinite sequences $(x_j)_j$ with $x_j - x_{j-1} \geqslant \Delta$ (thus for $\underline{\Delta} \geqslant 0$ in our terminology). In view of [7, Theorem 3] and Lemma 5, (i) $\Longleftrightarrow$ (iii) this would mean that $f \in \Lambda^p \cap R_{\mathrm{loc}}$ iff and only if $|f|^p$ is improperly Riemann integrable over $\mathbb{R}$ and belongs to the class $BCV(\mathbb{R})$.

Whereas the $BCV(\mathbb{R})$-condition is rather abstract (it is a larger class than $BV(\mathbb{R})$ and only meaningful for unbounded intervals, for otherwise BCV is equivalent to f being bounded) our condition $f \in \Lambda^p$ immediately implies that $\left\{\frac{1}{w} \sum_{j\in\mathbb{Z}} |f(j/w)|^p\right\}^{1/p} < \infty$, yielding that the operator S_w maps Λ^p into $L^p(\mathbb{R})$, see Theorem 7.

3.2. The approximate sampling theorem in $L^p(\mathbb{R})$-norm

In this section we consider sampling series $S_w f$ of functions $f \in \Lambda^p$ for $1 < p < \infty$. We regard Λ^p as a semi-normed linear space where the semi-norm $\|f\|_{l^p(\Sigma_w)}$ is that of Definition 4(b) based on the equidistant partition $\Sigma_w = \{\frac{k}{w}; k \in \mathbb{Z}\}$ for $w > 0$. The following result will be basic (see [5,1]).

Theorem 7. *For every $f \in \Lambda^p$, $1 < p < \infty$, there holds*

$$\|S_w f\|_{L^p(\mathbb{R})} \leqslant C \|f\|_{l^p(\Sigma_w)} \quad (w > 0),$$

where the constant C is independent of f and w.

This theorem states that the sampling series defines a linear operator from Λ^p to $L^p(\mathbb{R})$ uniformly bounded with respect to w.

The main result of this section is the proof of Theorem 2, namely that the classical sampling theorem (Theorem A) yields the approximate sampling theorem with respect to the L^p-norm, namely the counterpart of Theorem B.

Proof of Theorem 2. Let $\varepsilon > 0$. Since $\bigcup_{\tau>0} B^p_{\pi\tau}$ is dense in $L^p(\mathbb{R})$ for all $1 \leqslant p < \infty$ (see [14, Section 5.5][2]) there exists a $\tau > 0$ and exists a function $g \in B^p_{\pi\tau}$ such that $\|f - g\|_{L^p(\mathbb{R})} < \varepsilon$. Next choose $w_0 \geqslant \tau$ such that $\|f - g\|_{l^p(\Sigma_w)} < \|f - g\|_{L^p(\mathbb{R})} + \varepsilon$ for all $w > w_0$, which is possible in view of Lemma 6(b).

Then, according to the classical sampling theorem, $S_w g = g$ for $g \in B^p_{\pi\tau} \subset W^1(L^p(\mathbb{R})) \cap C(\mathbb{R}) \subset \Lambda^p$, and hence by Theorem 7,

$$\|S_w f - f\|_{L^p(\mathbb{R})} \leqslant \|S_w f - S_w g\|_{L^p(\mathbb{R})} + \|g - f\|_{L^p(\mathbb{R})} \leqslant C\|f - g\|_{l^p(\Sigma_w)} + \|g - f\|_{L^p(\mathbb{R})}.$$

[2] The density can be proved using, e.g., the singular convolution integral of de La Vallée Poussin $J(f;\rho) \in B^p_{2\rho}$ for $f \in L^p(\mathbb{R})$ satisfying $\lim_{\rho\to\infty} \|J(f;\rho) - f\|_{L^p(\mathbb{R})} = 0$; see [1, Section 5] or [14, Section 8.6].

Now, it follows from the choices of τ and w that

$$C\|f-g\|_{l^p(\Sigma_w)}+\|g-f\|_{L^p(\mathbb{R})}<C\left\{\|g-f\|_{L^p(\mathbb{R})}+\varepsilon\right\}+\|g-f\|_{L^p(\mathbb{R})}<C(\varepsilon+\varepsilon)+\varepsilon,$$

and we end up with

$$\|S_w f - f\|_{L^p(\mathbb{R})} < (2C+1)\varepsilon \quad (w > w_0). \qquad \square$$

An examination of this proof shows that it is essentially a Banach–Steinhaus-type argument. The uniform boundedness is established in Theorem 7, and the convergence on a dense subset holds in view of Theorem A.

Observe that whereas Theorem B includes Theorem A, the L^p-approximate sampling theorem (Theorem 2) does not include Theorem A. This is due to the fact that in the latter case there holds no error estimate in the form (3). Nevertheless, as to error estimates for the L^p-counterpart we have:

If f just belongs to Λ^p, $1 < p < \infty$, *then*

$$\|S_w f - f\|_{L^p(\mathbb{R})} \leqslant c_r \tau_r(f; w^{-1}; M(\mathbb{R}))_p \tag{20}$$

holds for any $r \in \mathbb{N}$.

Here $\tau_r(f; \Delta; M(\mathbb{R}))_p$ is the so-called L^p-averaged modulus of smoothness of $f \in M(\mathbb{R})$ of order $r \in \mathbb{N}$. The right-hand side of (20) can be estimated by $w^{-1}\omega_r(f'; w^{-1}; L^p(\mathbb{R}))$ provided $f \in W^1(L^p(\mathbb{R})) \cap C(\mathbb{R})$, ω_r being the classical L^p-modulus of continuity. See [1] for the details; the many examples treated there reveal that the approach also covers signals which can be badly discontinuous, in particular, those which have jump discontinuities which may even form a set of measure zero on $\mathbb{R}$.

The error estimate (20) with $r = 1$ can also be found in [17, Theorem 23], however, under the stronger condition that $f \in L^p(\mathbb{R})$ satisfies $\sup_{\Delta>0} \Delta^{-s}\tau_m(f; \Delta; M(\mathbb{R}))_p < \infty$ for some $0 < s < 1/p$ and some $m > s$, i.e., f must satisfy a Lipschitz condition of order s with respect to τ_m in order to fulfill (20) with $r = 1$. The proof is carried out using certain function spaces $A^s_{p,\infty}$. These spaces are defined in terms of the τ-modulus in a similar manner as the well-known Nikol'skiĭ spaces $B^s_{p,\infty}$ are defined by means of the classical ω-modulus. Apart from inequality (20) this paper contains some $\mathcal{O}$-estimates for the error $\|S_w f - f\|_{L^p(\mathbb{R})}$ which are deduced in the general setting of the Besov–Nikol'skiĭ space theory; such estimates can also be deduced from our inequality (20).

Indeed, estimate (20) is valid for arbitrary $r \in \mathbb{N}$ without any Lipschitz condition imposed upon f. The proof in [1] seems to be more elementary than that in [17]. It does not use any general theory and is essentially based on a Jackson-type inequality and an interpolation theorem for discrete operators.

4. The missing link between two groups of major equivalent theorems and formulae

The actual goal of the theorems presented is not so obvious. The classical sampling theorem belongs to a certain group of formulae, all members of which are equivalent in the sense that each can be deduced from any of the others by elementary means. This group includes the sampling theorem, Poisson's summation formula of Fourier analysis in the case of bandlimited functions, and Cauchy's integral formula of function theory for functions belonging to B^∞_σ.

The approximate (or generalized) sampling theorem, however, belongs to another group of formulae, which are again equivalent to each other. This group includes the approximate sampling theorem, the Poisson summation formula in the general case (for $f, f' \in L^1(\mathbb{R})$), the Euler–Maclaurin summation formula and the Abel–Plana summation formula, both of numerical analysis, as well as the functional equation of the Riemann zeta function of analytic number theory; see [4, Section 6.1; 3, p. 84; 8, pp. 90–96].

Now the theorems of this paper, especially Theorem 1, present the open, missing link between the foregoing two groups of major theorems and formulae, all members of which are equivalent to one another. This is due to the fact that the sampling theorem is fully equivalent (thus without further conditions being imposed) to the generalized sampling theorem in the uniform norm. This means that the two groupings presented are basically equivalent to one another, in particular eight theorems and formulae from different areas of analysis and number theory are essentially equivalent, each implying the others. Specifically, Poisson's summation formulae for bandlimited functions B^1_σ is equivalent to the formula in the general instance. The matter is connected with a fundamental conjecture raised by S. Bochner (1974), namely that "the Poisson summation formula and Cauchy's integral and residue formulas are two different aspects of a comprehensive broad-gauged duality formula, which lies athwart most of analysis".

The aim of the authors' ongoing work in this area is to examine in greater detail and precision the proofs of implication in many groupings of equivalent results that they have considered in recent years, groupings that go back to work of Hamburger in the early 1920s. For example, Mordell [13] showed that Poisson's summation formula yields the functional equation for the Riemann zeta function, and Ferrar [6] is said to have shown that the functional equation yields Poisson's formula; however, much work remains to be done in checking that all these proofs are simple and basic, use a minimum of resources and are, above all, free of circular reasoning.

A further aim of the papers to follow is the addition of further assertions equivalent to those of the two groupings, such as the famous partial fraction expansion of the cotangent function $\cot(\pi z)$, the Tschakalov sampling theorem, etc. For results in this direction see Higgins et al. [11,9,10].

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